# INVERSE PROBLEM OF THE VIBRATIONS OF RODS AND STRINGS OF VARIABLE MASS

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We formulate the inverse problem for longitudinal, torsional, and transverse vibrations of rods and transverse vibrations of strings of variable mass. It is shown that if certain assumptions are made concerning the densities of the combining and separating particles, the problem reduces to the integration of independent partial differential equations and a system of ordinary differential equations of first order.

1. Combining and separating particles may be of three types. The first type includes particles which combine and separate at each point of the surface of a rod or string and form a unified solid medium with the rod or string. The second type includes particles which combine or separate at each point of the surface of an elastic body and are connected with the rod (or string), interact with it, and do not interact with one another. Particles of the third type combine and separate at some discrete points of the rod or string.

The combining and separation of the particles does not affect the criteria of an elastic body which characterize it as a rod or string. The axis of the rod does not change its position with respect to the rod. At the moment of combining or separation, the particles move parallel to the displacement of the points of the rod (or string). The x axis is directed along the axis of the undeformed rod (or string) from its left end toward its right end. The length of the rod (or string) is denoted by $l$, and the transverse motion of its points is denoted by $z(x, t)$.

The rod is divided into segments by the points $M_i$ $(x_i)$ $(x_i < x_{i+1};\ x_1 = 0;\ x_n = l;\ i = 1, 2, \ldots, n)$, at which there are particles of the third type. If at the end points $M_1$ (0) and $M_n$ $(l)$ there are no such particles, we shall assume that such particles are there but have zero mass.

For particles of the first type, we introduce the following notation: $\rho_1^\circ$ (x) is the initial linear density at the point M (x); $\rho_1^k$ (x, t) and $v_{1a}^k$ (x, t) (k = 1, 2, 3, 4) are the linear densities and absolute velocities of of the combining and separating particles at time t. The superscript 1 indicates a combining particle and its movement in the direction of increasing z. The superscript 2 also indicates a combining particle, but one which moves in the direction of decreasing z. The superscript 3 indicates a separating particle moving in the positive z direction. The superscript 4 means that a particle is separating and moving in the direction of decreasing z. We have

$$\rho_1^k(x, 0) = 0, \frac{\partial \rho_1^1}{\partial t} \geqslant 0, \quad \frac{\partial \rho_1^2}{\partial t} \geqslant 0, \quad \frac{\partial \rho_1^3}{\partial t} \leqslant 0, \frac{\partial \rho_1^4}{\partial t} \leqslant 0,$$

$$\rho_1 = \rho_1^0 + \rho_1^1 + \rho_1^2 + \rho_1^3 + \rho_1^4$$

$$R_{1r} = \sum_{k=1}^{4} (\partial \rho_1^k / \partial t)\, v_{1r}^k$$

where $R_{1r}$ is the intensity of the reactive forces of the combining and separating particles in their relative motion; $v_{1r} = v_{1a}^k - \partial z/\partial t$ are the relative velocities of the particles.

This notation, except for the subscript, will be retained for the corresponding values associated with the other types of particles. The subscript for particles of the second type will be 2, and the subscript for

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particles of the third type will be $M_i$. In the expressions for values associated with particles of the third type, partial derivatives should be replaced with ordinary derivatives. For particles of the first and second types, the mass density and the intensity of reactive forces are the values for mass and force per unit length. For particles of the third type, these are the mass and force concentrated at the point in question. It will be convenient hereafter to regard the mass density and force intensity for particles of the third type as linear. Then the mass density and reactive-force intensity will take the form

$$\rho = \rho_1 + \rho_2 + \sum_{i=1}^{n} \rho_{M_i} \sigma_1 (x - x_i), \tag{1.1}$$

$$R_r = R_{1r} + R_{2r} + \sum_{i=1}^{n} R_{M_i r} \sigma_1 (x - x_i)$$

where $\sigma_1$ is a pulse function of first order.

The inverse problem for the vibrations of elastic bodies of variable mass will be formulated by analogy with the mechanics of discrete systems of variable mass [1]. For a given external loading Q(x, t) and a given law of vibration z(x, t), we are required to find the density of the body at any instant of time at any point of the body. The absolute or relative velocities of the combining and separating particles are given as functions of the coordinates and time. We also know $\rho_1^\circ$, $\rho_2^\circ$, $\rho_{M_i}^\circ$. In what follows, the formulation of the inverse problem will be made more precise by the introduction of additional conditions.

For boundary points the formulation of the inverse problem is the following. If we do not know the reactions of the external medium on the end points, then the masses $\rho_{M_1}$ and $\rho_{M_n}$ are given values. In this case we must determine the unknown reactions acting on the ends of the body. If the reactions are known, what we must determine are the masses of the end points.

The relation between the linear density and the displacements of the points of the rod is expressed by the differential equation for the vibrations of a rod of variable mass [2]. This differential equation involves $\rho_1^k$, $\rho_2^k$, $\rho_{M_i}^k$ and their derivatives with respect to time. Therefore, we must first determine these quantities and then determine the density of the body. The differential equation for the vibrations is not sufficient for finding $\rho_1^k$, $\rho_2^k$, $\rho_{M_i}^k$. We must also know additional conditions relating these quantities to one another. They may be of various kinds. We shall confine ourselves to the case in which these conditions are given in the form of algebraic relations among $\rho_1^k$, $\rho_2^k$, $\rho_{M_i}^k$, i.e., the densities of the combining and separating particles are connected by the relations

$$f_\alpha (\rho_1^1, \rho_1^2, \ldots, \rho_2^4, x, t) = 0, \qquad f_{i\beta} (\rho_{M_i}^1, \ldots, \rho_{M_i}^4, t) = 0 \tag{1.2}$$

$$\alpha = 1, 2, \ldots, 7; \ \beta = 1, 2, 3; \ i = 1, 2, \ldots, n$$

To these conditions we add the equations

$$f_8 (\rho_1^1, \rho_1^2, \ldots, \rho_2^4, x, t) = \mu, \qquad f_{i4} (\rho_{M_i}^1, \ldots, \rho_{M_i}^4, t) = m_i \tag{1.3}$$

Relations (1.2) and (1.3) are such that after solving them for $\rho_1^k$, $\rho_2^k$, $\rho_{M_i}^k$, we obtain the following functions, which are continuous and have continuous derivatives with respect to $\mu$ and t:

$$\rho_1^k = \varphi_1^k(\mu, x, t), \quad \rho_2^k = \varphi_2^k(\mu, x, t), \quad \rho_{M_i}^k = \varphi_{M_i}^k (m_i, t) \tag{1.4}$$

$$\varphi_1 = \varphi_1^\circ + \sum_{k=1}^{4} \varphi_1^k, \quad \varphi_2 = \rho_2^\circ + \sum_{k=1}^{4} \varphi_2^k, \quad \varphi_{M_i} = \rho_{M_i}^\circ + \sum_{k=1}^{4} \varphi_{M_i}^k$$

$$k = 1, 2, 3, 4; \qquad i = 1, 2, 3, 4$$

After differentiating $\varphi_1^k$, $\varphi_2^k$, $\varphi_{M_i}^k$ with respect to t and substituting the derivatives into the expression for the intensity of the reactive forces, we have

$$R_r = a \frac{\partial \mu}{\partial t} + g + \sum_{i=1}^{n} \left( a_i \frac{dm_i}{dt} + g_i \right) \sigma_1 (x - x_i) \tag{1.5}$$

$$a = \sum_{k=1}^{4} \left( v_{1r}^k \frac{\partial \varphi_1^k}{\partial \mu} + v_{2r}^k \frac{\partial \varphi_2^k}{\partial \mu} \right), \quad g = \sum_{k=1}^{4} \left( v_{1r}^k \frac{\partial \varphi_1^k}{\partial t} + v_{2r}^k \frac{\partial \varphi_2^k}{\partial t} \right) \tag{1.6}$$

$$a_i = \sum_{k=1}^{4} v_{M_i r}^k \frac{\partial \varphi_{M_i}^k}{\partial m_i}, \quad g_i = \sum_{k=1}^{4} v_{M_i r}^k \frac{\partial \varphi_{M_i}^k}{\partial t}, \quad i = 1, 2, \ldots, n$$

2. The equation for longitudinal vibrations of a rod of variable mass can be written in the form

$$(\rho_1 + \rho_2)\frac{\partial^2 z}{\partial t^2} + \sum_{i=1}^{n} \rho_{M_i} \frac{d^2 z(x_i, t)}{dt^2} \sigma_1(x - x_i) - \frac{\partial}{\partial x}\left(\varkappa \frac{\partial z}{\partial x}\right) = R_{1r} + R_{2r} + \sum_{i=1}^{n} R_{M_i r}\sigma_1(x - x_i) + Q \tag{2.1}$$

where $\varkappa$ is the rigity of the rod for longitudinal vibrations.

For torsional vibrations of a rod, the quantities pertaining to translational motion in (2.1) must be replaced with the analogous quantities for rotational motion. From physical considerations, we can conclude that for longitudinal and torsional vibrations of a rod, $\varkappa$ can be a function of $\rho_1$, x, and t. Taking account of the equations in (1.4), we find that the rigidity of the rod is $\varkappa = \psi(\mu, x, t)$.

We assume that all the functions involved in Eq. (2.1) are continuous with respect to t, except for $\varkappa(\partial z/\partial x)$. They are all continuous with respect to x. The function $\varkappa(\partial z/\partial x)$ has discontinuities of the first kind at the points $M_i$ and is continuous at all other points. If $M_i\varkappa(\partial t/\partial x)$ were continuous at the points $M_i$, the vibrations of the rod would not affect the motion of the mass $\rho_{M_i}$. We integrate Eq. (2.1) with respect to x between the limits $x_i - \varepsilon$ and $x_i + \varepsilon$, after which we let $\varepsilon$ approach zero. We obtain a differential equation for the vibrations of the mass $\rho_{M_i}$:

$$\rho_{M_i} \frac{d^2 z(x_i, t)}{dt^2} = \omega_i + R_{M_i r} + F_i, \qquad i = 1, 2, \ldots, n \tag{2.2}$$

where $\omega_i = [\psi\ (\partial z/\partial x)]_{x_i+0} - [\psi\ (\partial z/\partial x)]_{x_i-0}$ is the jump of $\varkappa(\partial z/\partial x)$ at the point $M_i$; $\omega_i$ takes account of the effect of the rod on the motion of the mass $\rho_{M_i}$; for the left end $\omega_1 = (\psi\,\partial z/\partial x)_{x=0}$, and for the right end $\omega_n = -(\psi\ \partial z/\partial x)_{x=l}$, since the rigidity outside the rod is zero, and $F_i$ is the concentrated external force applied to the mass $\rho_{M_i}$.

In (2.2) we replace $\rho_{M_i}$ and $R_{M_i r}$ with their expressions from (1.4) and (1.6) and thus obtain a system of n independent ordinary differential equations of first order for determining the $m_i$:

$$a_i \frac{dm_i}{dt} = c_i \qquad (i = 1, 2, \ldots, n) \tag{2.3}$$

In these formulas

$$c_i = \varphi_{M_i} \frac{d^2 z(x_i, t)}{dt^2} - \omega_i - \sum_{k=1}^{4} v_{M_i r}^{k} \frac{\partial \varphi_{M_i}^{k}}{\partial t} - F_i$$

For each of the segments $[x_i, x_{i+1}]$ ($i = 1, 2, \ldots n-1$) we have a partial differential equation for determining $\mu$ which was obtained from Eq. (2.1):

$$a\frac{\partial \mu}{\partial t} + b\frac{\partial \mu}{\partial x} = c \tag{2.4}$$

$$b = \frac{\partial z}{\partial x}\frac{\partial \psi}{\partial \mu}, \qquad c = (\varphi_1 + \varphi_2)\frac{\partial^2 z}{\partial t^2} - \frac{\partial z}{\partial x}\frac{\partial \psi}{\partial x} - \psi\frac{\partial^2 z}{\partial x^2} - \sum_{k=1}^{4}\left(v_{1r}{}^{k}\frac{\partial \varphi_1{}^{k}}{\partial t} + v_{2r}{}^{k}\frac{\partial \varphi_2{}^{k}}{\partial t}\right) - Q$$

This equation is a quasilinear partial differential equation of first order. The theory and methods of solution of such equations are known [3]. In what follows it will be assumed that the coefficients $a$, b, c satisfy requirements such that for all $x_i \le x \le x_{i+1}$, $0 \le t \le T_i$ ($i = 1, 2, \ldots, n-1$) the solution exists and is unique. The initial and boundary conditions must be specified. The initial condition is obtained from (1.3) for $t = 0$. In order to specify the boundary condition, we must know $\mu = f_8$ from (1.3) as a function of t at one of the ends of the segment $[x_i, x_{i+1}]$. If $f_8$ is known for the left end of the segment $[x_i, x_{i+1}]$, we must take this as the value of $\mu$ immediately to the right of the point $x_i$. If $f_8$ is known for the right end of the segment, then the value of $\mu$ is given immediately to the left of the point $x_{i+1}$ The boundary condition is given in such a way (i.e., the selected end of the segment $[x_i, x_{i+1}]$ is selected in such a way) that the characteristics of Eq. (2.4) in the plane of variable x and t will not intersect one another and will have at most one point in common with the curve on which the initial and boundary conditions are given.

Specifying $f_8$ as a function of time at one end of the segment $[x_i, x_{i+1}]$ is a feature which must be added in the formulation of the inverse problem.

The sequence of steps gone through in solving the inverse problem is the following: from Eq. (2.4) we must find $\mu$; next, from (1.4) we must determine $\rho_1$ and $\rho_2$ and express $\varkappa$ in terms of x and t. After this, we solve Eq. (2.3) for the $m_i$, and from (1.4) we find the $\rho_{M_i}$. Using formula (1.1), we determine the linear density. In the case in which we have to find the reaction acting at one of the ends, we must use the corresponding equation of (2.3). This equation will be algebraic in the unknown reaction.

Equations (2.3) and (2.4) can be algebraic if the $a_1$, $a$, and b are identically zero. This may happen, for example, if the relative velocities of the combining and separating particles are zero, and $\psi$ does not depend explicitly on $\mu$. In the case when only one of the coefficients $a$, b is identically zero, Eq. (2.4) can be regarded as an ordinary differential equation, where we treat as a parameter the variable with respect to which there is no partial differentiation in the equation. If $a \equiv 0$, the initial condition is unnecessary; if $b \equiv 0$, the boundary condition is unnecessary.

Let us consider an example. A rod with a fixed end at $x=0$ and a free end at $x=l$ undergoes free longitudinal vibrations according to the law $z=u \sin t$, where $u=\varepsilon x$ for $0 \leq x \leq l/2$ and $u=\varepsilon(l-x)$ for $l/2 \leq x \leq l$ ($\varepsilon$ is a small positive constant). Particles of the first type combine with the rod. We have $\rho_1^\circ=\text{const}$, $\rho_1^1=\rho_1^2$, $\rho_2^3=\rho_1^4=0$, $\varkappa=2\varepsilon^{-2}\rho_1$. For $x=l/2$ the rigidity varies according to the law $\varkappa=2\varepsilon^{-2}\rho_1^\circ \exp[{}^1/_2[\varepsilon l+2t)t]$. The relative velocities of the combining particles are such that $v_{1r}^1+v_{1r}^2)=-\sin t$.

Variable masses $\rho_{M_2}$ and $\rho_{M_3}$ are affixed to the midpoint of the rod and its free end. The relative velocities of the combining and separating particles for the mass $\rho_{M_2}$ are zero. Particles are separating from mass $\rho_{M_3}$, so that $\rho_{M_3}{}^1=\rho_{M_3}{}^2=0$, $\rho_{M_3}{}^3=\rho_{M_4}{}^4$, ${}^1/_2\,(v_{M_3r}{}^3+v_{M_3r}{}^4)=-\varepsilon^{-1}\sin t$.

Setting $\rho_1=\mu$, we write Eq. (2.4) as

$$-\frac{\partial\mu}{\partial t}+2\frac{\partial\mu}{\partial u}+\mu u=0$$

Solving this equation, we find $\rho_1=\rho_1^\circ \exp[(u+t)t]$. From the first equation of the system (2.3), which is an algebraic equation, we find the reaction at the fixed left end of the rod, $F_r=-2\varepsilon^{-1}\rho_1^\circ \sin t \exp t^2$.

The equation for determining the mass $\rho_M$ is also an algebraic equation. Setting $c_2=0$, we obtain $\rho_{M_2}=4\varepsilon^{-2}l^{-1}\rho_1^\circ \exp[{}^1/_2\,(\varepsilon l+2t)\,t]$.

The third equation of the system Eq. (2.3) is a differential equation. Setting $m_3=\rho_{M_3}$ in this equation, we obtain

$$\rho_{M_3}=\rho^\circ_{M_3}-2\rho_1^\circ\int_0^t \exp(\xi^2)\,d\xi$$

3. On the basis of the analogy between the longitudinal vibrations of a rod and the transverse vibrations of a string, we can assert that the differential equation of the vibrations of a string of variable mass coincides with Eq. (2.1). The tension of the string, like the rigidity in the case of longitudinal vibrations of a rod, may depend on $\rho_1$, x, and t. Making use of Eq. (1.4), we obtain $\varkappa=\psi(\mu, x, t)$. According to [4], the tension at each instant of time is identical for all points of the string, i.e., $\psi(\mu(x, t), x, t)$ is a function which, if not constant, varies only with time. Consequently,

$$\frac{\partial\psi}{\partial\mu}\frac{\partial\mu}{\partial x}+\frac{\partial\psi}{\partial x}=0 \tag{3.1}$$

This relation enables us to reduce the differential equation (2.4) of the inverse problem of the vibrations of a string of variable mass on the segment $[x_i, x_{i+1}]$ to the form

$$a\frac{\partial\mu}{\partial t}=c_1 \tag{3.2}$$

where $a$ is found from formula (1.5) and

$$c_1=(\varphi_1+\varphi_2)\frac{\partial^2 z}{\partial t^2}-\psi\frac{\partial^2 z}{\partial x^2}-\sum_{k=1}^{4}\left(v_{1r}{}^k\frac{\partial\varphi_1{}^k}{\partial t}+v_{2r}{}^k\frac{\partial\varphi_2{}^k}{\partial t}\right)-Q$$

Equation (3.2) is an ordinary differential equation involving the coordinate x as a parameter, and therefore it is unnecessary to give any boundary conditions.

The differential equations of the inverse problem for the masses $\rho_{M_i}$ are the same as (2.3), but the $\omega_i$ in these equations can be represented in the form

$$\omega_i = \psi\left[\left(\frac{\partial z}{\partial x}\right)_{x_i+0} - \left(\frac{\partial t}{\partial x}\right)_{x_i-0}\right]$$

This equation means that when the masses $\rho_{M_i}$ interact with the string at the points $M_i$ ($i = 2, 3, \ldots, n-1$) the condition of smoothness of the string is violated, i.e., at these points the string has discontinuities.

To the case in which $\psi$ does not depend explicitly on x there corresponds the solution of Eq. (3.2), which is a function of t. This means that $c_1$ and $a$ are also functions of time. This condition imposes certain limitations on the relative velocities of the combining and separating particles and their densities.

If at least one end of the string is free, then $\psi(\mu, x, t) \equiv 0$. We may assume that the form of the function $\psi$ is unknown. After determining $\mu$ as a function of $f(x, t)$ from (3.2), it is sufficient to set $\psi = \mu - f(x, t)$. If the form of the function $\psi$ is known, then from the equation $\psi = 0$ we find $\mu$ as a function of x and t. This function must satisfy two requirements. Firstly, when it is substituted into Eq. (3.2), the equation must become an identity; secondly, this function must satisfy the initial conditions.

The sequence of steps followed in solving the inverse problem for the vibrations of a string of variable mass is the same as the sequence of steps for the analogous problem involving the longitudinal vibrations of a rod of variable mass.

Let us consider an example. A string whose ends are fixed undergoes free vibrations according to the law $z = \varepsilon \sin(\pi x/l) \sin t$, where $\varepsilon$ is a small positive constant. Particles of the first type combine with the string; $\rho = \rho_1$, $\rho_1^\circ = \text{const}$, $\rho_1^1 = \rho_1^2$, $\rho_1^3 = \rho_1^4 = 0$, $1/2\,(v_{1r}^1 + v_{1r}^2) = z$, $\varkappa = 2\pi^{-2} l^2 \rho$. Setting $\mu = \rho$, we can write Eq. (3.2) in the form $d\mu/dt = \mu$. From this we find $\rho = \rho_1^\circ e^t$. The equations in (2.3) enable us to find the reactions at the fixed ends. These reactions are equal to $2\varepsilon l \pi^{-1} \rho_1^\circ e^t \sin t$.

4. We write the differential equation for the transverse vibrations of a rod of variable mass [2]:

$$(\rho_1 + \rho_2)\frac{\partial^2 z}{\partial t^2} + \sum_{i=1}^{n} \rho_{M_i} \frac{d^2 z(x_i, t)}{dt^2} \sigma_1(x - x_i) + \frac{\partial^2}{\partial x^2}\left(\varkappa \frac{\partial^2 z}{\partial x^2}\right) = R_{1r} + R_{2r} + \sum_{i=1}^{n} R_{M_i r} \sigma_1(x - x_i) + Q \qquad (4.1)$$

where $\varkappa(\rho_1, x, t)$ is the rigidity of the rod for deflection.

All the functions occurring in Eq. (4.1) are continuous with respect to t and $\mu$. All of them except $(\partial/\partial x)[\varkappa(\partial^2 z/\partial x^2)]$ are continuous with respect to x. The function $(\partial/\partial x)[\varkappa(\partial^2 z/\partial x^2)]$ has discontinuities of the first kind at the points $x_i$ and is continuous at all other points. On each of the segments $[x_i, x_{i+1}]$ ($i = 1, 2, \ldots, n-1$) we have a partial differential equation for determining $\mu$, found from Eq. (3.1):

$$a\frac{\partial \mu}{\partial t} = p\frac{\partial^2 \mu}{\partial x^2} + q\left(\frac{\partial \mu}{\partial x}\right)^2 + r\frac{\partial \mu}{\partial x} + c \qquad (4.2)$$

The coefficient $a$ is found from formula (1.5). The other coefficients have the form

$$p = \frac{\partial \psi}{\partial \mu}\frac{\partial^2 z}{\partial x^2}, \quad q = \frac{\partial^2 \psi}{\partial \mu^2}\frac{\partial^2 z}{\partial x^2}, \quad r = 2\left(\frac{\partial \psi}{\partial \mu}\frac{\partial^3 z}{\partial x^3} + \frac{\partial^2 \psi}{\partial x \partial \mu}\frac{\partial^2 z}{\partial x^2}\right)$$

$$c = (\varphi_1 + \varphi_2)\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 \psi}{\partial x^2}\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial \psi}{\partial x}\frac{\partial^3 z}{\partial x^3} + \psi\frac{\partial^4 z}{\partial x^4} - \sum_{k=1}^{4}\left(v_{1r}^k \frac{\partial \varphi_1^k}{\partial t} + v_{2r}^k \frac{\partial \varphi_2^k}{\partial t}\right) - Q$$

Equation (4.2) is a nonlinear partial differential equation of second order [5]. Its coefficients $a$, p, q, r, c are assumed to be such that the hypothesis of the existence and uniqueness theorem for the mixed problem when $x_i \le x \le x_{i+1}$, $0 \le t \le T$, is satisfied. In addition to the initial condition obtained from (1.3), we specify the boundary conditions as follows. We shall assume that in (1.3) $f_8$ is a function of time immediately to the right of the point $x_i$ and immediately to the left of the point $x_{i+1}$. This is the additional condition that must be introduced into the formulation of the inverse problem involving transverse vibrations of a rod of variable mass.

The differential equations of motion of the masses $\rho_{M_i}$ have the same form as the equations in (2.3), but in these equations the $\omega_i$ are different from the case of longitudinal vibrations. For transverse vibrations the $\omega_i$ are given by the formula

$$\omega_i = \left[\frac{\partial}{\partial x}\left(\varkappa \frac{\partial^2 z}{\partial x^2}\right)\right]_{x_i-0} - \left[\frac{\partial}{\partial x}\left(\varkappa \frac{\partial^2 z}{\partial x^2}\right)\right]_{x_i+0}, \quad i = 1, 2, \ldots, n$$

Equations (4.2) and (2.3) can be solved in any sequence, since the $\omega_0$ can be found by using the boundary conditions.

Let us consider an example. A rod with free ends at $x=0$ and $x=l$ undergoes free transverse vibrations according to the law $z = \varepsilon x(l-x)\sin t$, $\varepsilon=\text{const}>0$. Particles of the first kind separate from the rod; $\rho_1^1=\rho_1^2=0$, $\rho_1^3=\rho_1^4$, $\rho_1^\circ=\rho x$, $\beta=\text{const}>0$, $\rho(0,t)=0$, $\rho(l,t)=\beta l e^{-\varepsilon t}$. The rigidity of the rod varies according to the law $\varkappa=\alpha\rho_1$, $\alpha=\text{const}>0$. The relative velocities of the separating particles are such that $\frac{1}{2}(v_{II}^3+v_{II}^4)=x(l-x)\sin t$. Setting $\mu=\rho=\rho_1$, we write Eq. (3.2) in the form

$$x(l-x)\frac{\partial\mu}{\partial t}+2\alpha\varepsilon\frac{\partial^2\mu}{\partial x^2}+\varepsilon x(l-x)\mu=0$$

The solution of this equation, which is obtained by separation of variables, is $\rho=\beta x e^{-\varepsilon t}$.

5. In our earlier discussion we distinguished between particles of the first and second types which combine with or separate from the rod or string; the reason for this distinction was that particles of the first type form a unified solid medium with the rod or string, whereas in the case of particles of the second type the internal stresses are so small that they may be neglected.

Inverse problems in which there are no combining or separating particles of the first type are of no interest, since Eqs. (2.4) and (4.2) become ordinary differential equations of first order with the coordinate x as a parameter. It is unnecessary to specify the boundary conditions. The reason for this is that combination and separation of particles of the first type affects the variation of the rigidity of the rod, with the result that Eqs. (2.4) and (4.2) contain derivatives of $\mu$ with respect to x. Therefore, as boundary conditions, we can specify the rigidity as a function of time at the ends of the segment $[x_i, x_{i+1}]$. In other words, as the function $f_8$ in (1.3) we should take an appropriate function of $\varkappa$, x, and t. In particular, if $f_8$ is linear with respect to $\varkappa$, then the coefficient q in Eq. (4.2) vanishes.

In the solution of inverse problems of longitudinal and transverse vibrations of rods, the initial and boundary conditions for Eqs. (2.4) and (3.2) must satisfy consistency conditions. These conditions determine the degree of smoothness of the problem.

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